

Amplitudes of mono-components and representation by generalized sampling functions

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Abstract

A mono-component is a real-valued signal of finite energy that has non-negative instantaneous frequencies, which may be defined as the derivative of the phase function of the given real-valued signal through the approach of canonical amplitude-phase modulation. We study in this article how the amplitude is determined by its phase in a canonical amplitude-phase modulation. Our finding is that such an amplitude can be perfectly reconstructed by a sampling formula using the so-called generalized sampling functions and their Hilbert transforms. The regularity of such an amplitude is identified to be at least continuous. Meanwhile, we also make a very interesting and new characterization of the band-limited functions.

Keywords: mono-component, generalized sampling function, analytic signal, nonlinear phase, amplitude-phase modulation, Hilbert transform, Blaschke product, Poisson kernel.

1 Introduction

Any real-valued non-stationary signal f of finite energy, that is, f is in the space $L^2(\mathbb{R})$ of square integrable functions on the set \mathbb{R} of real numbers, may be represented as an *amplitude-phase modulation* with a time-varying amplitude ρ and a time-varying phase ϕ where phase ϕ is, in general, *nonlinear*. Specifically, the value of f at $t \in \mathbb{R}$ may be represented as

$$f(t) = \rho(t) \cos \phi(t). \quad (1.1)$$

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Unfortunately, this type of representation is not unique because the modulation is obtained through a complex signal that can have various choices of the imaginary part. However, one can determine a unique such factorization (1.1) by using the approach of analytic signals. Indeed, let $\mathcal{A}(f)$ be the *analytic signal associated with f* with the characteristic property

$$(\mathcal{A}(f))^\wedge(\omega) = \begin{cases} 2\hat{f}(\omega) & \text{if } \omega \geq 0 \\ 0 & \text{if } \omega < 0, \end{cases} \quad (1.2)$$

where for any signal $g \in L^2(\mathbb{R})$, $\hat{g} = \mathcal{F}g$ is the Fourier transform of g defined at $\xi \in \mathbb{R}$ by the equation

$$\hat{g}(\xi) = (\mathcal{F}g)(\xi) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} g(t) e^{-i\xi t} dt. \quad (1.3)$$

Equation (1.2) is equivalent to for $t \in \mathbb{R}$,

$$\mathcal{A}(f)(t) = f(t) + i\mathcal{H}f(t),$$

where the operator $\mathcal{H} : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ stands for the Hilbert transform, and for $f \in L^2(\mathbb{R})$, $\mathcal{H}f$ at $t \in \mathbb{R}$ is defined through the principal value integral

$$\mathcal{H}f(t) := \text{p.v.} \frac{1}{\pi} \int_{\mathbb{R}} \frac{f(x)}{t-x} dx = \lim_{\epsilon \rightarrow 0} \int_{|x-t| > \epsilon} \frac{f(x)}{t-x} dx.$$

The value $\mathcal{A}(f)(t)$ at $t \in \mathbb{R}$ is complex which can be written into the *quadrature form*

$$\mathcal{A}(f)(t) = \rho(t) e^{i\phi(t)}.$$

Under the conditions that the derivative value $\phi'(t)$ is non-negative or non-positive for all $t \in \mathbb{R}$, the quantities $\rho(t)$ and $\phi'(t)$ are called the *instantaneous amplitude* and *instantaneous frequency* at $t \in \mathbb{R}$, of the real signal f , respectively. The corresponding modulation (1.1) is then called the *canonical amplitude-phase modulation*, or *canonical modulation* for short. The signal f with such defined non-negative instantaneous frequencies is thus called a *mono-component*. A large body of literature addresses this problem, see for example, [1, 2, 8, 11, 12, 9, 16].

Constructing the canonical pair (ρ, ϕ) of the instantaneous amplitude and phase is important in the theories of analytic signals. It is equivalent to the problem of seeking the function pair (ρ, ϕ) such that for $t \in \mathbb{R}$, the following equation holds true

$$\mathcal{H}(\rho(\cdot) \cos \phi(\cdot))(t) = \rho(t) \sin \phi(t). \quad (1.4)$$

We remark that (1.4) can be apparently considered as a special case of the *Bedrosian identity*

$$\mathcal{H}(fg) = f\mathcal{H}(g).$$

In [1], the author proved that, if both f, g belong to $L^2(\mathbb{R})$, f is of lower frequency, g is of higher frequency and f, g have no overlapping frequency, then $\mathcal{H}(fg) = f\mathcal{H}(g)$. This classic

result of Bedrosian is not useful for constructing a mono-component. The reason lies in that the requirement of both f and g in $L^2(\mathbb{R})$ is invalid.

Recently, an important phase function that renders mono-components was given in [13]. The phase function is defined through the boundary values of a Blaschke product on a unit disk $\Delta := \{z : z \in \mathbb{C}, |z| \leq 1\}$, where \mathbb{C} indicates the set of complex numbers. Specifically, for $a \in (-1, 1)$, the Blaschke product at $z \in \mathbb{C} \setminus \{1/a\}$ is given by

$$B_a(z) = \frac{z - a}{1 - az}. \quad (1.5)$$

Subsequently the *non-linear phase* function, denoted by θ_a , is defined at $t \in \mathbb{R}$ by the equation

$$e^{\theta_a(t)} := B_a(e^{it}). \quad (1.6)$$

If we recall that the periodic Poisson kernel p_a whose value at $t \in \mathbb{R}$ is given by

$$p_a(t) := \frac{1 - a^2}{1 - 2a \cos t + a^2}, \quad (1.7)$$

then by taking the derivative of both sides of equation (1.6), we find that the phase θ_a is an anti-derivative of p_a , and its derivative is always positive, that is,

$$\frac{d}{dt} \theta_a(t) = p_a(t) > 0.$$

We shall in this paper characterize the amplitude function ρ of finite energy when the phase function ϕ is chosen at $t \in \mathbb{R}$ by

$$\phi(t) = \theta_a(t) = \int_{[0,t]} p_a(x) dx$$

such that equation (1.4) is satisfied. Our main result indicates that such kind of amplitude can be perfectly reconstructed in terms of a sampling formula using the *generalized sampling function* whose value at $t \in \mathbb{R}$ is given by

$$\text{sinc}_a(t) := \frac{\sin \theta_a(t)}{t}. \quad (1.8)$$

In Section 2, we review the construction of the generalized sampling function and discuss some properties pertained to it. In Section 3, we introduce the concept of Bedrosian subspace of the Hilbert transform and investigate some properties of functions in this space. In Section 4, we make an important observation when a *linear phase* is chosen, the amplitude function must be bandlimited in order to satisfy equation (1.4). In Section 5, we present our main result in Theorem 5.7.

2 Generalized sampling functions

Not very surprisingly the function sinc_a has many properties that are similar to the classic *sinc* defined at $t \in \mathbb{R}$ by the equation

$$\text{sinc}(t) := \frac{\sin t}{t}.$$

Those properties include cardinality, orthogonality, decaying rate, among others. In the special case $a = 0$, the function sinc_a reduces to the classic sinc, which will become clear later. Let us first review the approach to obtain an explicit form of sinc_a .

The classic sinc function is fundamentally significant in digital signal processing due to the Shannon sampling theorem [14, 15, 3]. The Shannon sampling theorem enables to reconstruct a bandlimited signal from shifts of sinc functions weighted by the uniformly spaced samples of that signal. Recently efforts have been made to extend the classic sinc to generalized sampling functions, for example, in [5, 6, 7]. Intuitively, the spectrum of the sinc function is just the indicator function of a symmetric interval of finite measure. Hence, authors in [5] are inspired to consider functions with piecewise polynomial spectra to replace the usual sinc function for the purpose of sampling non-bandlimited signals. One kind of generalized sampling functions given in [5], denoted by sinc_a that is related to a constant $a \in (-1, 1)$, is defined as the *inverse Fourier transform* of a so-called *symmetric cascade filter*, denoted by H_a . Specifically,

$$\text{sinc}_a := \sqrt{\frac{\pi}{2}}(1+a)\mathcal{F}^{-1}H_a. \quad (2.9)$$

Let \mathbb{N} be the set of natural numbers, \mathbb{Z} be the set of integers, and $\mathbb{Z}_+ := \{0\} \cup \mathbb{N}$. Let X be a subset of \mathbb{R} , and for $q \in \mathbb{N}$, we say a function f is in $L^q(X)$ if and only if the $L^q(X)$ norm

$$\|f\|_{q,X} := \left(\int_X |f(t)|^q dt \right)^{1/q} < \infty.$$

Similarly, let Z be a subset of \mathbb{Z} , a sequence $\mathbf{y} := (y_k : k \in Z)$ is said to be in $l^q(Z)$ if and only if the $l^q(Z)$ norm

$$\|\mathbf{y}\|_{q,Z} := \left(\sum_{k \in Z} |y_k|^q \right)^{1/q} < \infty.$$

The symmetric cascade filter H_a is a piecewise constant function whose value at $\xi \in \mathbb{R}$ is given by

$$H_a(\xi) := \sum_{n \in \mathbb{Z}_+} a^n \chi_{I_n}(\xi), \quad (2.10)$$

where χ_I is the *indicator function* of the set I , and the interval I_n , $n \in \mathbb{Z}_+$, is the union of two symmetric intervals given by the equation

$$I_n := (-(n+1), -n] \cup [n, (n+1)).$$

Of course, we have that $H \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ because the sequence $\mathbf{b} := (a^n : n \in \mathbb{Z}_+) \in l^1(\mathbb{Z}_+) \cap l^2(\mathbb{Z}_+)$, and hence $\text{sinc}_a \in L^2(\mathbb{R})$ since the Fourier operator is closed in $L^2(\mathbb{R})$ and sinc_a is continuous because $H_a \in L^1(\mathbb{R})$.

The symmetric cascade filter H_a can be associated with an *analytic function* F on the open unit disk Δ defined at $z \in \Delta$ by

$$F(z) := \sum_{n \in \mathbb{Z}_+} a^n z^n. \quad (2.11)$$

Thus, by substituting equation (2.10) into equation (2.9) and making use of equation (2.11) an alternative form of $\text{sinc}_a(t)$, $t \in \mathbb{R}$ in terms of F can be found as

$$\text{sinc}_a(t) = (1+a) \text{sinc} \left(\frac{t}{2} \right) \text{Re} \left\{ F(e^{it}) e^{\frac{1}{2}it} \right\}, \quad (2.12)$$

where $\text{Re}(z)$ is the real part of a complex number z .

A very interesting fact, as discovered in the paper [5], is that the function F is linked to the Blaschke Product B_a by the equation

$$(1+a)F(z) := \frac{B_a(z) - 1}{z - 1}. \quad (2.13)$$

Plugging the formula (2.13) into equation (2.12) we readily obtain the explicit expression of sinc_a given earlier in equation (1.8).

The next two formulas shall be used later. Expanding the left-hand side of equation (1.6) using Euler's formula and separating the real part from the imaginary part of the right-hand side, one obtains that for $t \in \mathbb{R}$

$$\sin \theta_a(t) = \frac{(1-a^2) \sin t}{1-2a \cos t + a^2} = p_a(t) \sin t \quad (2.14)$$

and

$$\cos \theta_a(t) = \frac{(1+a^2) \cos t - 2a}{1-2a \cos t + a^2}. \quad (2.15)$$

We next list some properties of the function sinc_a .

Proposition 2.1 *Let the generalized sampling function sinc_a be defined by equation (2.12) or equation (1.8). Then the following statements hold.*

1. For $t \in \mathbb{R}$,

$$\text{sinc}_a(t) = \frac{(1-a^2)}{1-2a \cos t + a^2} \frac{\sin t}{t} = p_a(t) \text{sinc}(t). \quad (2.16)$$

2.

$$\mathcal{F} \text{sinc}_a = \sqrt{\frac{\pi}{2}} (1+a) H_a. \quad (2.17)$$

3. $\text{sinc}_a(n\pi) = \frac{1+a}{1-a}\delta_{n,0}$, where $\delta_{n,0} = 1$ if $n = 0$ and $\delta_{n,0} = 0$ if $n \in \mathbb{Z} \setminus \{0\}$.
4. sinc_a is even, bounded, infinitely differentiable.
5. $|\text{sinc}_a(t)| \leq \frac{1+a}{1-a} \frac{2}{1+|t|}$ for $t \in \mathbb{R}$, and $\text{sinc}_a \in L^2(\mathbb{R})$.
6. The set $\{\text{sinc}_a(\cdot - n\pi) : n \in \mathbb{Z}\}$ is an orthogonal set, that is

$$\langle \text{sinc}_a, \text{sinc}_a(\cdot - n\pi) \rangle = \frac{(1+a)}{1-a} \pi \delta_{n,0},$$

where $\langle u, v \rangle = \int_{\mathbb{R}} u(t) v^*(t) dt$ denotes the usual inner product of the two functions $u, v \in L^2(\mathbb{R})$, and v^* is the complex conjugate of v .

Proof: Equation (2.16) directly follows by substituting equation (2.14) into equation (1.8). Equation (2.17) is obtained by taking the inverse Fourier transform of both sides of equation (2.9). The third and fourth statements directly follow from equation (2.16). The fifth statement follows from equation (2.16) and noticing $\text{sinc}(t) \leq \frac{2}{1+|t|}$ and $p_a(t) \leq \frac{1+a}{1-a}$ for any $t \in \mathbb{R}$. The last statement is a special case of Corollary 3.2 of [5]. For the convenience of readers, we provide a direct proof here. By Parseval's theorem and equation (2.17) we have

$$\begin{aligned} \int_{\mathbb{R}} \text{sinc}_a(t) \text{sinc}_a(t - n\pi) dt &= \frac{\pi}{2} (1+a)^2 \int_{\mathbb{R}} H_a^2(x) e^{in\pi x} dx \\ &= \frac{\pi}{2} (1+a)^2 \int_{\mathbb{R}} \sum_{k \in \mathbb{Z}_+} a^{2k} \chi_{I_k}(x) e^{in\pi x} dx \\ &= \frac{\pi}{2} (1+a)^2 \sum_{k \in \mathbb{Z}_+} a^{2k} \int_{I_k} e^{in\pi x} dx = \frac{1+a}{1-a} \pi \delta_{n,0}, \end{aligned}$$

where, in the last equality we have used the orthogonality identity $\int_{I_k} e^{in\pi x} dx = 2\delta_{n,0}$, $k \in \mathbb{Z}_+$. The interchange of the integral operator and the infinite sum is guaranteed by the absolute convergence of the series. \square

In figure 2.1, we show an example of the Fourier transform pair sinc_a and $\sqrt{\frac{\pi}{2}}(1+a)H_a$ with $a = 0.5$. In the plot of sinc_a , the graph of the standard sinc is also shown, which corresponds to the case $a = 0$.

Next we review several basic properties of the Hilbert transform which we will need frequently later. These properties can be found, for example, in the book [10]. First the Hilbert transform is an *anti-involution*, that is,

$$\mathcal{H}^2 = -\mathcal{I} \tag{2.18}$$

where \mathcal{I} is the identity operator. Second the operator \mathcal{H} is anti-self adjoint, that is,

$$\langle \mathcal{H}u, v \rangle = \langle u, -\mathcal{H}v \rangle. \tag{2.19}$$

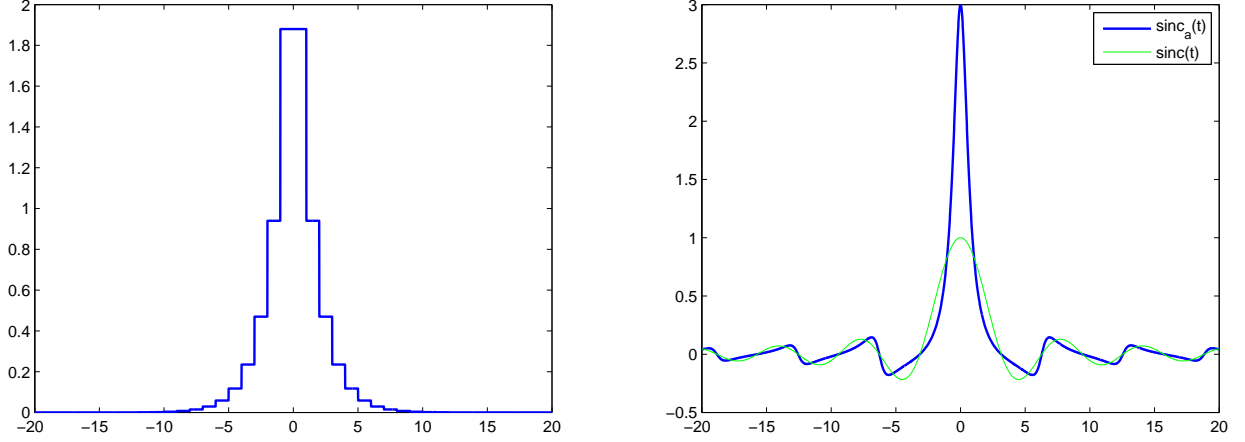


Figure 2.1: Left: the graph of $\sqrt{\frac{\pi}{2}}(1+a)H_a$, $a = 0.5$; Right: the graph of sinc_a , $a = 0.5$.

Third, for any $f \in L^2(\mathbb{R})$ and $t \in \mathbb{R}$, the composition of the Fourier transform and the Hilbert transform is given by

$$\mathcal{F}(\mathcal{H}f)(t) = -i \operatorname{sgn}(t) \mathcal{F}f(t), \quad (2.20)$$

where $\operatorname{sgn}(\cdot)$ is the *signum function* having values defined by $\operatorname{sgn}(x) = 1$ if $x \in \mathbb{R}_+ := \{t \in \mathbb{R} : t > 0\}$, $\operatorname{sgn}(x) = -1$ if $x \in \mathbb{R}_- := \{t \in \mathbb{R} : t < 0\}$, and $\operatorname{sgn}(0) = 0$.

Theorem 2.2 *The system*

$$\Phi := \{\text{sinc}_a(\cdot - 2k\pi), \mathcal{H}\text{sinc}_a(\cdot - 2k\pi) : k \in \mathbb{Z}\} \quad (2.21)$$

is an orthogonal system in $L^2(\mathbb{R})$.

Proof: By the third statement of Proposition 2.1, we have that

$$\langle \text{sinc}_a, \text{sinc}_a(\cdot - 2k\pi) \rangle = \frac{1+a}{1-a} \delta_{0k}.$$

Invoking equations (2.18) and (2.19) yields

$$\begin{aligned} \langle \mathcal{H}\text{sinc}_a, \mathcal{H}\text{sinc}_a(\cdot - 2k\pi) \rangle &= \langle \text{sinc}_a, -\mathcal{H}^2 \text{sinc}_a(\cdot - 2k\pi) \rangle \\ &= \langle \text{sinc}_a, \text{sinc}_a(\cdot - 2k\pi) \rangle = \frac{1+a}{1-a} \delta_{0k}. \end{aligned}$$

By Parseval's theorem and equation (2.17) we have

$$\int_{\mathbb{R}} \text{sinc}_a(t) \mathcal{H}\text{sinc}_a(t - 2k\pi) dt = \frac{(1+a)^2 \pi i}{2} \int_{\mathbb{R}} H_a^2(t) \operatorname{sgn}(t) e^{i2k\pi t} dt. \quad (2.22)$$

Invoking the expression (2.10) of H_a , equation (2.22) becomes

$$\begin{aligned}
& \int_{\mathbb{R}} \text{sinc}_a(t) \mathcal{H} \text{sinc}_a(t - 2k\pi) dt \\
&= \frac{(1+a)^2 \pi i}{2} \int_{\mathbb{R}} \sum_{n \in \mathbb{Z}_+} a^{2n} \chi_{I_n}(t) \text{sgn}(t) e^{i2k\pi t} dt \\
&= \frac{(1+a)^2 \pi i}{2} \sum_{n \in \mathbb{Z}_+} a^{2n} \int_{I_n} \text{sgn}(t) e^{i2k\pi t} dt \\
&= \frac{(1+a)^2 \pi i}{2} \sum_{n \in \mathbb{Z}_+} a^{2n} \left[\int_{[n, n+1)} e^{i2k\pi t} dt - \int_{(-(n+1), -n]} e^{i2k\pi t} dt \right],
\end{aligned}$$

where the interchange of the integral and the infinite sum in the second equality is guaranteed by the absolute convergence of the series.

When $k = 0$, the difference in the pair of brackets is zero, while when $k \in \mathbb{Z} \setminus \{0\}$, each integral inside the pair of brackets is zero. Therefore we have

$$\langle \text{sinc}_a, (\mathcal{H} \text{sinc}_a)(\cdot - 2k\pi) \rangle = 0$$

for any $k \in \mathbb{Z}$.

□

3 Bedrosian subspace of the Hilbert transform

In this section, we pay attention to the set

$$\mathcal{S}_a := \{f : f \in L^2(\mathbb{R}), \mathcal{H}(f \cos \theta_a(\cdot)) = f \sin \theta_a(\cdot)\}. \quad (3.23)$$

It is clear that the set \mathcal{S}_a is a subspace of $L^2(\mathbb{R})$ due to the linearity of the Hilbert transform. The subspace \mathcal{S}_a shall be called the *Bedrosian subspace* of the Hilbert transform. We first make a simple observation that we shall need frequently later.

Lemma 3.1 *Equation (1.4) is true if and only if for $t \in \mathbb{R}$,*

$$\mathcal{H}(\rho(\cdot) e^{i\phi \cdot})(t) = -i\rho(t) e^{i\phi t}. \quad (3.24)$$

Proof: Applying the Hilbert transform \mathcal{H} to both sides of equation (1.4) and utilizing equation (2.18) yields

$$\mathcal{H}(\rho(\cdot) \sin(\phi \cdot))(t) = -\rho(t) \cos(\phi t). \quad (3.25)$$

Combining equations (1.4) and (3.25) produces equation (3.24). Thus equation (1.4) is equivalent to equation (3.24). □

Lemmas 3.2–3.4 are several technical lemmas we need in the sequel.

Lemma 3.2 Suppose that $f \in L^2(\mathbb{R})$ and $g \in L^2(-\frac{\tau}{2}, \frac{\tau}{2})$ is a τ -periodic function having Fourier series $g(t) = \sum_{k \in \mathbb{Z}} c_k e^{ik\frac{2\pi}{\tau}t}$ with $c_k = \frac{1}{\tau} \int_{(-\frac{\tau}{2}, \frac{\tau}{2})} g(t) e^{-i\frac{2\pi}{\tau}kt} dt$. Moreover, the series $\sum_{k \in \mathbb{Z}} c_k$ is absolutely convergent. Then $fg \in L^2(\mathbb{R})$ and its Fourier transform at $\xi \in \mathbb{R}$ is given by

$$(fg)^\wedge(\xi) = \sum_{k \in \mathbb{Z}} c_k \hat{f}\left(\xi - \frac{2\pi}{\tau}k\right).$$

Proof: The fact that $g \in L^2(-\frac{\tau}{2}, \frac{\tau}{2})$ implies that $|g(t)| \leq c$, for a.e. $t \in \mathbb{R}$ and some constant c . Thus $\int_{\mathbb{R}} f^2(t)g^2(t) dt \leq c^2 \int_{\mathbb{R}} f^2(t) dt < \infty$. That is, $fg \in L^2(\mathbb{R})$. By the definition (1.3) of the Fourier transform, we have

$$\begin{aligned} (fg)^\wedge(\xi) &= \int_{\mathbb{R}} f(t) \left(\sum_{k \in \mathbb{Z}} c_k e^{ik\frac{2\pi}{\tau}t} \right) e^{-it\xi} dt \\ &= \sum_{k \in \mathbb{Z}} c_k \int_{\mathbb{R}} f(t) e^{-it(\xi - k\frac{2\pi}{\tau})} dt \\ &= \sum_{k \in \mathbb{Z}} c_k \hat{f}\left(\xi - \frac{2\pi}{\tau}k\right), \end{aligned}$$

where the interchange of the integral and the infinite sum is guaranteed by the absolute convergence of the series $\sum_{k \in \mathbb{Z}} c_k$. \square

We say that a complex τ -periodic signal g is *circularly analytic* if its Fourier expansion is the one-sided series $g(t) = \sum_{k \in \mathbb{Z}_+} c_k e^{i\frac{2\pi}{\tau}kt}$, $t \in \mathbb{R}$.

Lemma 3.3 Suppose that $f \in L^2(\mathbb{R})$ is a real-valued function and the complex function g is τ -periodic and circularly analytic with real Fourier coefficients c_k , $k \in \mathbb{Z}_+$. Then

$$\mathcal{H}(fg)(t) = -if(t)g(t) \quad (3.26)$$

if and only if for a.e. $\xi \in \mathbb{R}_-$,

$$\sum_{k \in \mathbb{Z}_+} c_k \hat{f}\left(\xi - \frac{2\pi}{\tau}k\right) = 0. \quad (3.27)$$

Proof: Equation (3.26) indicates that fg is an analytic signal. This is equivalent to say that fg has no negative frequency, that is, $(fg)^\wedge(\xi) = 0$, $\xi \in \mathbb{R}_-$. By Lemma 3.2 and the fact that g is circularly analytic, we obtain equation (3.27). \square

Lemma 3.4 The 2π -periodic function $g := e^{i\theta_a(\cdot)}$ is circularly analytic and the Fourier expansion of $e^{i\theta_a(\cdot)}$ at $t \in \mathbb{R}$ is given by

$$e^{i\theta_a(t)} = -a + \sum_{k \in \mathbb{N}} (1 - a^2) a^{k-1} e^{ikt}. \quad (3.28)$$

Proof: Recalling equation (1.6), the Fourier coefficients c_k , $k \in \mathbb{Z}$, of g is given by

$$c_k = \frac{1}{2\pi} \int_{(-\pi, \pi)} \frac{e^{it} - a}{1 - ae^{it}} e^{-ikt} dt.$$

Let $\text{Rez}(f(z); c)$ indicate the residue of the function f at the point $c \in \mathbb{C}$. Denote the boundary of the unit disc Δ by $\partial\Delta$. For $k = 0$, we have $c_0 = \frac{1}{2\pi i} \oint_{\partial\Delta} \frac{z-a}{1-az} \frac{dz}{z} = \text{Res} \left(\frac{z-a}{(1-az)z}; 0 \right) = -a$. In the case k is a negative integer, by using the Cauchy theorem we get that

$$c_k = \frac{1}{2\pi} \int_{(-\pi, \pi)} \frac{e^{it} - a}{1 - ae^{it}} e^{i|k|t} dt = \frac{1}{2\pi i} \oint_{\partial\Delta} \frac{(z-a)z^{|k|-1}}{1-az} dz = 0,$$

because the integrand is analytic in the unit disk Δ as $|a| < 1$. We are left to consider the case of $k \in \mathbb{N}$. Using the residue theorem and the formula for $j \in \mathbb{N}$,

$$\frac{d^j}{dz^j} \left(\frac{z-a}{1-az} \right) = (1-a^2)a^{j-1}j!(1-az)^{-j-1},$$

leads to when $k \in \mathbb{N}$,

$$\begin{aligned} c_k &= \frac{1}{2\pi i} \oint_{\partial\Delta} \frac{z-a}{(1-az)z^{k+1}} dz = \text{Res} \left(\frac{z-a}{(1-az)z^{k+1}}; 0 \right) \\ &= \frac{1}{k!} \lim_{z \rightarrow 0} \frac{d^k}{dz^k} \left(\frac{z-a}{1-az} \right) = (1-a^2)a^{k-1}. \end{aligned}$$

The proof of this lemma is completed. □

Next proposition indicates that the subspace \mathcal{S}_a is *invariant* under the Hilbert transform.

Proposition 3.5 *Suppose that $\rho \in \mathcal{S}_a$, then $\mathcal{H}\rho \in \mathcal{S}_a$.*

Proof: We first claim that the complex-valued signal $(\rho + i\mathcal{H}\rho)e^{i\theta_a}$ is an analytic signal. Indeed, by Lemma 3.4 the signal $e^{i\theta_a}$ is circularly analytic, and we write $e^{i\theta_a(t)} = \sum_{k \in \mathbb{Z}_+} g_k e^{ikt}$. Then by Lemma 3.2, we have for $\xi \in \mathbb{R}$,

$$\mathcal{F}((\rho + i\mathcal{H}\rho)e^{i\theta_a})(\xi) = \sum_{k \in \mathbb{Z}_+} g_k \mathcal{F}(\rho + i\mathcal{H}\rho)(\xi - k). \quad (3.29)$$

Note that $(\rho + i\mathcal{H}\rho)$ is an analytic signal thus it has no negative spectrum, that is $\mathcal{F}(\rho + i\mathcal{H}\rho)(\xi) = 0$ when $\xi \in \mathbb{R}_-$. Consequently, when $\xi \in \mathbb{R}_-$,

$$\sum_{k \in \mathbb{Z}_+} g_k \mathcal{F}(\rho + i\mathcal{H}\rho)(\xi - k) = 0,$$

which implies by Lemma 3.3 that

$$\mathcal{H}[(\rho + i\mathcal{H}\rho)e^{i\theta_a}] = -i(\rho + i\mathcal{H}\rho)e^{i\theta_a}.$$

Simplifying both sides leads to $\mathcal{H}[(\mathcal{H}\rho)e^{i\theta_a}] = -i[(\mathcal{H}\rho)e^{i\theta_a}]$, which indicates that $\mathcal{H}\rho \in \mathcal{S}_a$. \square

The next lemma indicates that any real-valued function $\rho \in \mathcal{S}_a$ is completely determined by its spectrum $\hat{\rho}\chi_{(-1,1)}$ on the interval $(-1, 1)$.

Lemma 3.6 *The real-valued signal $\rho \in \mathcal{S}_a$ if and only if for a.e. $\xi \in \mathbb{R} \setminus \{0\}$, $n \in \mathbb{Z}_+$,*

$$\hat{\rho}(\xi + \text{sgn}(\xi)n) = a^n \hat{\rho}(\xi). \quad (3.30)$$

Proof: By Lemma 3.1 $\rho \in \mathcal{S}_a$ is equivalent to for $t \in \mathbb{R}$,

$$\mathcal{H}(\rho(\cdot)e^{i\theta_a(\cdot)})(t) = -i\rho(t)e^{i\theta_a(t)}, \quad (3.31)$$

which is equivalent to saying that the signal $\rho(\cdot)e^{i\theta_a(\cdot)}$ is an analytic signal. Applying Lemmas 3.3 and 3.4 to the signal $\rho(\cdot)e^{i\theta_a(\cdot)}$ yields that $\rho \in \mathcal{S}_a$ holds if and only if for a.e. $\xi \in \mathbb{R}_-$,

$$a\hat{\rho}(\xi) = (1 - a^2) \sum_{k \in \mathbb{N}} a^{k-1} \hat{\rho}(\xi - k). \quad (3.32)$$

In equation (3.32), separating the first term from the rest in the sum on the right-hand side, we have

$$\begin{aligned} a\hat{\rho}(\xi) &= (1 - a^2) \left(\hat{\rho}(\xi - 1) + \sum_{k \in 1+\mathbb{N}} a^{k-1} \hat{\rho}(\xi - k) \right) \\ &= \hat{\rho}(\xi - 1) - a^2 \hat{\rho}(\xi - 1) + (1 - a^2) \sum_{k \in 1+\mathbb{N}} a^{k-1} \hat{\rho}(\xi - k). \end{aligned} \quad (3.33)$$

On the other hand, multiplying both sides of equation (3.32) by a and replacing ξ by $\xi - 1$ we also have

$$\begin{aligned} a^2 \hat{\rho}(\xi - 1) &= (1 - a^2) \sum_{k \in \mathbb{N}} a^k \hat{\rho}(\xi - 1 - k) \\ &= (1 - a^2) \sum_{k \in 1+\mathbb{N}} a^{k-1} \hat{\rho}(\xi - k). \end{aligned} \quad (3.34)$$

Combining equations (3.33) and (3.34) deduces that for a.e. $\xi \in \mathbb{R}_-$,

$$\hat{\rho}(\xi - 1) = a\hat{\rho}(\xi).$$

By induction, we can get that for $n \in \mathbb{Z}_+$ and a.e. $\xi \in \mathbb{R}_-$,

$$\hat{\rho}(\xi - n) = a^n \hat{\rho}(\xi). \quad (3.35)$$

Reversing the above calculations reveals that condition (3.35) is also sufficient for (3.32).

Taking the conjugate of both sides of equation (3.32) and using the Hermitian property of $\hat{\rho}$, we obtain an equivalent equation

$$a\hat{\rho}(\xi) = (1 - a^2) \sum_{k \in \mathbb{N}} a^{k-1} \hat{\rho}(\xi + k) \quad (3.36)$$

for a.e. $\xi \in \mathbb{R}_-$. A similar discussion to the case $\xi \in \mathbb{R}_-$ gives that equation (3.36) is equivalent to

$$\hat{\rho}(\xi + n) = a^n \hat{\rho}(\xi) \quad (3.37)$$

for $n \in \mathbb{Z}_+$ and a.e. $\xi \in \mathbb{R}_-$. Finally combining equations (3.35) and (3.37) produces (3.30). \square

Proposition 3.7

$$\text{sinc}_a \in \mathcal{S}_a \quad \text{and} \quad \mathcal{H}\text{sinc}_a \in \mathcal{S}_a.$$

Proof: By Lemma 3.5, it suffices to show $\text{sinc}_a \in \mathcal{S}_a$. Observing the identity $H_a(\cdot + n) = a^{|n|} H_a(\cdot)$ for a.e. $\xi \in \mathbb{R} \setminus \{0\}$, $n \in \mathbb{Z}$ and recall equation (2.17) we thus have

$$\mathcal{F}(\text{sinc}_a)(\xi + |n|\text{sgn}(\xi)) = a^{|n|} \mathcal{F}(\text{sinc}_a)(\xi).$$

Hence by Lemma 3.6, we conclude that $\text{sinc}_a \in \mathcal{S}_a$. \square

4 An observation—how a linear phase determines the amplitude

In this section, we specifically consider the case when the phase ϕ in equation (1.4) is a linear phase and investigate the representation of the corresponding amplitude. The result is given in the following theorem. We use the notation $\text{supp}(f)$ for the set of real numbers on which the values $f(x)$ at $x \in \mathbb{R}$ are nonzero.

Theorem 4.1 *Suppose that γ is a positive real number and ρ is a non-zero real signal in $L^2(\mathbb{R})$. Then the following equation*

$$\mathcal{H}(\rho(\cdot) \cos(\gamma \cdot))(t) = \rho(t) \sin(\gamma t) \quad (4.1)$$

holds if and only if ρ is bandlimited with $\text{supp}(\hat{\rho}) \subset [-\gamma, \gamma]$.

Proof: Lemma 3.1 implies that equation (4.1) is equivalent to

$$\mathcal{H}(\rho(\cdot)e^{i\gamma\cdot})(t) = -i\rho(t)e^{i\gamma t}, \quad (4.2)$$

or equivalently, the complex signal $\rho(t)e^{i\gamma t}$ is an analytic signal. Note that the Fourier transform of the function on the left-hand side of equation (4.2) is given by $-i \operatorname{sgn}(\xi)\hat{\rho}(\xi - \gamma)$ and the Fourier transform of the right-hand side of equation (4.2) is $-i\hat{\rho}(\xi - \gamma)$. This leads to the equivalent equation of (4.1) in the frequency domain

$$(\operatorname{sgn}(\xi) - 1)\hat{\rho}(\xi - \gamma) = 0, \quad (4.3)$$

i.e.

$$\hat{\rho}(\xi) = 0, \quad \text{for } \xi \in (-\infty, -\gamma]. \quad (4.4)$$

Thus if ρ is bandlimited with $\operatorname{supp} \hat{\rho} \subset [\gamma, \gamma]$, then clearly equation (4.4) is true, hence equation (4.1) is true. On the other hand, assuming that equation (4.1) is true, we obtain that $\hat{\rho}(\xi) = 0$ for $\xi \in (-\infty, -\gamma]$. Since ρ is real-valued, by the Hermitian property of the Fourier transform of a real signal, $\hat{\rho}(\xi) = 0$ for $\xi \in [\gamma, +\infty)$. Consequently we conclude that ρ is a bandlimited function with its support in the frequency domain belongs to $[-\gamma, \gamma]$. \square

Remark: From this theorem we know that the amplitude function ρ can be represented by shifts of sinc function if and only if equation (4.1) is true.

5 How does non-linear phase determine amplitude?

In this section, we shall completely characterize a real-valued function $\rho \in \mathcal{S}_a$. We begin with introducing two one-sided filters that are related to the two-sided symmetric cascade filter H_a :

$$H_a^+(t) := H_a(t)\chi_{\mathbb{R}_+}(t), \quad H_a^-(t) := H_a(t)\chi_{\mathbb{R}_-}(t), \quad (5.1)$$

where $t \in \mathbb{R}$. The next lemma provides the Fourier transform pairs of H_a^+ and H_a^- , respectively.

Lemma 5.1 *The Fourier transform of H_a^+ and H_a^- are given, respectively, by*

$$(H_a^+(\cdot))^\wedge(\xi) = \frac{1}{\sqrt{2\pi}} \frac{1}{1 - ae^{-i\xi}} \frac{1 - e^{-i\xi}}{i\xi}, \quad (5.2)$$

and

$$(H_a^-(\cdot))^\wedge(\xi) = \frac{1}{\sqrt{2\pi}} \frac{1}{1 - ae^{i\xi}} \frac{1 - e^{i\xi}}{-i\xi}.$$

Proof: We start with the observation

$$H_a^+(t) = \sum_{k \in \mathbb{Z}_+} a^k \chi_{[k, k+1)}(t) = (1-a) \sum_{k \in \mathbb{N}} a^{k-1} \chi_{[0, k)}(t).$$

By the definition of the Fourier transform (1.3), we obtain that

$$\begin{aligned} (H_a^+(\cdot))^\wedge(\xi) &= \frac{1-a}{\sqrt{2\pi}} \int_{\mathbb{R}} \sum_{k \in \mathbb{N}} a^{k-1} \chi_{[0, k)}(t) e^{-i\xi t} dt \\ &= \frac{1-a}{\sqrt{2\pi}} \sum_{k \in \mathbb{N}} a^{k-1} \int_{[0, k)} e^{-i\xi t} dt \\ &= \frac{1-a}{\sqrt{2\pi}} \sum_{k \in \mathbb{N}} a^{k-1} \frac{e^{-ik\xi} - 1}{-i\xi}, \end{aligned}$$

where, the interchange of the order of the summation and the integral is justified by the absolute convergence of the series. Continue by noting $\sum_{k \in \mathbb{N}} a^{k-1} e^{ik\xi} = \frac{e^{i\xi}}{1-ae^{i\xi}}$ we have

$$\begin{aligned} (H_a^+(\cdot))^\wedge(\xi) &= \frac{1}{\sqrt{2\pi}} \frac{1 - ae^{-i\xi} - (1-a)e^{-i\xi}}{i\xi(1 - ae^{-i\xi})} \\ &= \frac{1}{\sqrt{2\pi}} \frac{1}{1 - ae^{-i\xi}} \frac{1 - e^{-i\xi}}{i\xi}. \end{aligned}$$

Using the identity $H_a^-(\cdot) = H_a^+(-\cdot)$, we obtain that

$$(H_a^-(\cdot))^\wedge(\xi) = (H_a^+(\cdot))^\wedge(-\xi) = \frac{1}{\sqrt{2\pi}} \frac{1}{1 - ae^{i\xi}} \frac{1 - e^{i\xi}}{-i\xi}.$$

□

Lemma 5.2 *The quasi-Bedrosian type identity*

$$\mathcal{H}(p_a(\cdot)\text{sinc}(\cdot))(t) = \frac{1+a}{1-a} p_a(t) \mathcal{H}\text{sinc}(t) \quad (5.3)$$

is true for $t \in \mathbb{R}$ and

$$\frac{1}{1 - ae^{it}} \frac{e^{it} - 1}{it} = \frac{1}{1+a} (\text{sinc}_a(t) + i\mathcal{H}\text{sinc}_a(t)). \quad (5.4)$$

Proof: Let $r(t) := \frac{1}{1-ae^{it}} \frac{e^{it}-1}{it}$. Separating the real part from the imaginary part of $r(t)$ produces

$$\begin{aligned} r(t) &= \frac{1-a}{1-2a\cos t + a^2} \frac{\sin t}{t} + i \frac{1+a}{1-2a\cos t + a^2} \frac{1 - \cos t}{t} \\ &= \frac{1}{1+a} p_a(t) \text{sinc}(t) + i \frac{1}{1-a} p_a(t) \mathcal{H}\text{sinc}(t), \end{aligned} \quad (5.5)$$

where we have used the identity

$$\mathcal{H}\text{sinc}(t) = \frac{1 - \cos t}{t}. \quad (5.6)$$

On the other hand, observe that $r(t) = \sqrt{2\pi} (H_a^+)^{\wedge}(-t)$, so the Fourier transform $\hat{r}(t) = \sqrt{2\pi} H_a^+(t)$ has zero negative spectrum. This implies r is an analytic signal. Therefore the imaginary part of $r(t)$ equals to the Hilbert transform of its real part, that is,

$$\mathcal{H} \left(\frac{1}{1+a} p_a(\cdot) \text{sinc}(\cdot) \right) (t) = \frac{1}{1-a} p_a(t) \mathcal{H}\text{sinc}(t).$$

After rearranging the above equation we obtain equation (5.3). Now we turn to show equation (5.4). Applying equation (5.3) to the imaginary part of equation (5.5) we continue to have that

$$\begin{aligned} r(t) &= \frac{1}{1+a} (p_a(t) \text{sinc}(t) + i \mathcal{H}(p_a(\cdot) \text{sinc}(\cdot))) (t) \\ &= \frac{1}{1+a} (\text{sinc}_a(t) + i \mathcal{H}\text{sinc}_a(t)), \end{aligned}$$

which is equation (5.4). \square

Corollary 5.3 *Let $\text{cosinc}_a(t) := \frac{1 - \cos \theta_a(t)}{t}$ and define $\text{cosinc}_a(0) = \lim_{t \rightarrow 0} \frac{1 - \cos \theta_a(t)}{t}$, then for $t \in \mathbb{R}$,*

$$\mathcal{H}\text{sinc}_a(t) = \text{cosinc}_a(t). \quad (5.7)$$

Moreover, the function cosinc_a is odd, bounded, infinitely differentiable, $\text{cosinc}_a(t) \leq \left(\frac{1+a}{1-a}\right)^2 \frac{3}{|t|+1}$ for $t \in \mathbb{R}$, and $\text{cosinc}_a \in L^2(\mathbb{R})$.

Proof: Equation (5.7) is just a new form of equation (5.3). Indeed, the left-hand side of equation (5.3) is just $\mathcal{H}\text{sinc}_a(t)$. Utilizing equations (1.7) and (5.9) then the right-hand side of equation (5.3) is simplified to $\text{cosinc}_a(t)$. The second statement follows by rewriting cosinc_a as

$$\text{cosinc}_a(t) = \frac{1+a}{1-a} p_a(t) \frac{1 - \cos t}{t}, \quad (5.8)$$

and noting the function

$$\mathcal{H}\text{sinc}(t) = \frac{1 - \cos t}{t} \quad (5.9)$$

is infinitely differentiable if we define $\mathcal{H}\text{sinc}(0) = \lim_{t \rightarrow 0} \frac{1 - \cos t}{t}$, and $\left| \frac{1 - \cos t}{t} \right| \leq \frac{3}{1+|t|}$ for any $t \in \mathbb{R}$. \square

An example of the graph of cosinc_a with $a = 0.5$ is shown in figure 5.2. As a comparison, the graph of $\mathcal{H}\text{sinc}(t) = \frac{1 - \cos t}{t}$ is also shown in the figure.

The following lemma appeared in [4]. However, a completely new proof by direct construction is given here for both sufficiency and necessity. The construction approach casts a new insight to understand the relation of the spectrum $\hat{\rho}$ of a function $\rho \in \mathcal{S}_a$ to the symmetric cascade filter H_a .

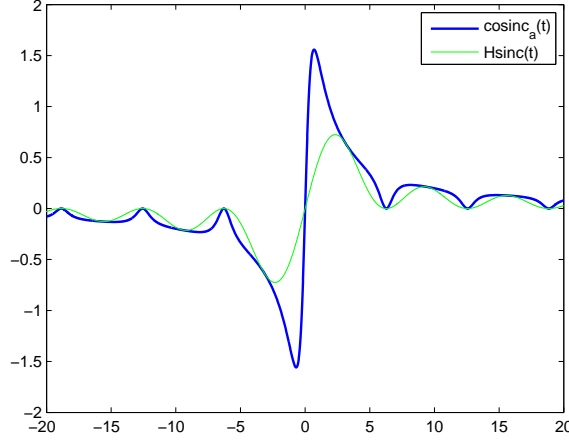


Figure 5.2: The graph of cosinc_a , $a = 0.5$.

Lemma 5.4 *A real signal $\rho \in \mathcal{S}_a$ if and only if there are two real sequences $\mathbf{r} = \{r_k : k \in \mathbb{Z}\}$ and $\mathbf{s} = \{s_k : k \in \mathbb{Z}\}$ in $l^2(\mathbb{Z})$ such that for $t \in \mathbb{R}$,*

$$\rho(t) = \sum_{k \in \mathbb{Z}} r_k \text{sinc}_a(t - 2k\pi) + \sum_{k \in \mathbb{Z}} s_k (\mathcal{H}\text{sinc}_a)(t - 2k\pi). \quad (5.10)$$

Proof: We first show the necessity, that is, if $\rho \in \mathcal{S}_a$, then ρ is given by equation (5.10). From Lemma 3.6, we know that $\rho(\xi)\chi_{\mathbb{R}_+}(\xi)$ and $\rho(\xi)\chi_{\mathbb{R}_-}(\xi)$ are completely determined by $\rho(\xi)\chi_{(0,1)}(\xi)$ and $\rho(\xi)\chi_{(-1,0)}(\xi)$, respectively. Expand $\rho(\xi)\chi_{(0,1)}(\xi)$ and $\rho(\xi)\chi_{(-1,0)}(\xi)$ into 1-periodic functions $\gamma_1(\xi) := \sum_{k \in \mathbb{Z}} u_k e^{i2k\pi\xi}$ and $\gamma_2(\xi) := \sum_{k \in \mathbb{Z}} v_k e^{i2k\pi\xi}$, where the two series $(u_k : k \in \mathbb{Z})$ and $(v_k : k \in \mathbb{Z})$ are in $l^2(\mathbb{Z})$.

By Lemma 3.6 we thus obtain that for a.e. $\xi \in \mathbb{R}$,

$$\hat{\rho}(\xi) = \gamma_1(\xi)H_a^+(\xi) + \gamma_2(\xi)H_a^-(\xi).$$

The Hermitian property of $\hat{\rho}$ implies that for $\xi \in \mathbb{R} \setminus \{0\}$,

$$\gamma_2(\xi) = (\gamma_1(-\xi))^*.$$

For convenience let $\gamma = \gamma_1$. Consequently $\hat{\rho}$ has the representation

$$\hat{\rho}(\xi) = \gamma(\xi)H_a^+(\xi) + (\gamma(-\xi))^* H_a^-(\xi),$$

for a.e. $\xi \in \mathbb{R}$. Let ρ_A be the analytic signal associated with ρ , which may be defined by

$$\hat{\rho}_A(\xi) = 2\hat{\rho}(\xi)\chi_{\mathbb{R}_+}(\xi)$$

or equivalently for $t \in \mathbb{R}$,

$$\rho_A(t) = \rho(t) + i\mathcal{H}\rho(t).$$

Then we have $\rho(t) = \text{Re}(\rho_A(t))$. We need to investigate ρ_A . The inverse Fourier transform of $\hat{\rho}_A$ yields

$$\begin{aligned}\rho_A(t) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} 2\gamma(\xi) H_a^+(\xi) e^{i\xi t} d\xi \\ &= \frac{2}{\sqrt{2\pi}} \int_{\mathbb{R}} \sum_{k \in \mathbb{Z}} u_k e^{-i2k\pi\xi} H_a^+(\xi) e^{i\xi t} d\xi.\end{aligned}$$

Applying the Lebesgue dominated convergence theorem to interchange the order of the integral and the sum, and appealing to equation (5.2) we obtain that

$$\begin{aligned}\rho_A(t) &= \frac{2}{\sqrt{2\pi}} \sum_{k \in \mathbb{Z}} u_k \int_{\mathbb{R}} H_a^+(\xi) e^{i\xi(t-2k\pi)} d\xi \\ &= \frac{2}{\sqrt{2\pi}} \sum_{k \in \mathbb{Z}} u_k \frac{1}{1 - ae^{i(t-2k\pi)}} \frac{e^{i(t-2k\pi)} - 1}{i(t-2k\pi)}.\end{aligned}$$

Denote $u_k = u_k^{(1)} + iu_k^{(2)}$. Recalling equation (5.4) we readily obtain that

$$\rho(t) = \frac{2}{\sqrt{2\pi}(1+a)} \left(\sum_{k \in \mathbb{Z}} u_k^{(1)} \text{sinc}_a(t-2k\pi) - \sum_{k \in \mathbb{Z}} u_k^{(2)} (\mathcal{H}\text{sinc}_a(\cdot))(t-2k\pi) \right).$$

Let $r_k = \frac{2}{\sqrt{2\pi}(1+a)} u_k^{(1)}$ and $s_k = -\frac{2}{\sqrt{2\pi}(1+a)} u_k^{(2)}$. Since the series $(u_k, k \in \mathbb{Z}) \in l^2(\mathbb{Z})$, we must have both series $(r_k, k \in \mathbb{Z})$ and $(s_k, k \in \mathbb{Z})$ in $l^2(\mathbb{Z})$. We have arrived at equation (5.10).

We now turn to the proof of sufficiency. It suffices to check that a function $\rho \in L^2(\mathbb{R})$ having the form (5.10) satisfies the equation (3.30) by Lemma 3.6. Applying the Fourier transform to both sides of equation (5.10) and noting that the Fourier transform of the generalized sampling function given by equation (2.17) we get that

$$\hat{\rho}(\xi) = H_a(\xi) (M(\xi) - i \text{sgn}(\xi) G(\xi))$$

with the two 1-periodic functions

$$M(\xi) = \frac{\sqrt{2\pi}}{2}(1+a) \sum_{k \in \mathbb{Z}} r_k e^{-i2k\pi\xi}, \quad G(\xi) = \frac{\sqrt{2\pi}}{2}(1+a) \sum_{k \in \mathbb{Z}} s_k e^{-i2k\pi\xi}.$$

Observing the identity $H_a(\cdot + n) = a^{|n|} H_a(\cdot)$ and the 1-periodicity of M and G lead to that, for $\xi \in \mathbb{R}_+$,

$$\begin{aligned}\hat{\rho}(\xi + |n|) &= H_a(\xi + |n|) (M(\xi + |n|) - iG(\xi + |n|)) \\ &= a^{|n|} H_a(\xi) (M(\xi) - iG(\xi)) = a^{|n|} \hat{\rho}(\xi).\end{aligned}$$

The case with negative ξ can be shown similarly. This completes the proof of the lemma. \square

Theorem 5.5 *If $\rho \in \mathcal{S}_a$, then both ρ and $\mathcal{H}\rho$ are continuous.*

Proof: By Proposition 3.5, $\mathcal{H}\rho \in \mathcal{S}_a$, thus it suffices to show $\rho \in \mathcal{S}_a$ is continuous. Indeed, if $\rho \in \mathcal{S}_a$, then ρ has the representation (5.10) by Lemma 5.4. Since both sinc_a and $\mathcal{H}\text{sinc}_a$ are in $L^2(\mathbb{R})$ by Proposition 2.1 and Corollary 5.3, consequently by the Cauchy-Schwarz inequality, the two series on the right-hand side of equation (5.10) converge uniformly, hence the limiting function ρ is continuous. \square

Lemma 5.6 *If $f \in \mathcal{S}_a$, then for $k \in \mathbb{Z}$,*

$$\langle f, \text{sinc}_a(\cdot - 2k\pi) \rangle = f(2k\pi), \quad \text{and} \quad \langle f, \mathcal{H}\text{sinc}_a(\cdot - 2k\pi) \rangle = -\mathcal{H}f(2k\pi).$$

Proof: The 2π -periodicity of $\sin \theta_a$ yields

$$\langle f, \text{sinc}_a(\cdot - 2k\pi) \rangle = \int_{\mathbb{R}} f(t) \frac{\sin \theta_a(t - 2k\pi)}{t - 2k\pi} dt = -\mathcal{H}(f \sin \theta_a)(2k\pi).$$

Therefor by the assumption on f , we have when $k \in \mathbb{Z}$,

$$-\mathcal{H}(f \sin \theta_a)(2k\pi) = f(2k\pi) \cos \theta_a(2k\pi) = f(2k\pi)$$

because $\cos \theta_a(2k\pi) = \cos \theta_a(0) = 1$ by equation (2.15).

The second equality follows by noting the fact that $\mathcal{H}f \in \mathcal{S}_a$ due to Proposition 3.5 and the Hilbert transformer is an anti-self adjoint operator thus

$$\langle f, \mathcal{H}\text{sinc}_a(\cdot - 2k\pi) \rangle = -\langle \mathcal{H}f, \text{sinc}_a(\cdot - 2k\pi) \rangle = -\mathcal{H}f(2k\pi).$$

\square

Lemma 5.4, Lemma 5.6, and Theorem 2.2 immediately implies the following theorem.

Theorem 5.7 *Any real-valued function $\rho \in \mathcal{S}_a$ if and only if*

$$\rho(t) = \frac{1-a}{1+a} \sum_{k \in \mathbb{Z}} \rho(2k\pi) \text{sinc}_a(t - 2k\pi) - \frac{1-a}{1+a} \sum_{k \in \mathbb{Z}} \mathcal{H}\rho(2k\pi) \mathcal{H}\text{sinc}_a(t - 2k\pi).$$

Moreover, the sampling sequences $(\rho(2k\pi), k \in \mathbb{Z})$ and $(\mathcal{H}\rho(2k\pi), k \in \mathbb{Z})$ are in $l^2(\mathbb{Z})$.

From Theorem 5.7 and Theorem 2.2 we have the following corollary.

Corollary 5.8 *The system Φ defined in (2.21) is a complete orthogonal system of the subspace $\mathcal{S}_a \subset L^2(\mathbb{R})$. That is, the subspace equals to the closure of the span of the set Φ , or symbolically,*

$$\mathcal{S}_a = \overline{\text{span}\Phi}.$$

Lastly, we state some facts for the case $a = 0$. Note when $a = 0$, $\sin \theta_a(t)$ becomes $\sin t$, $\cos \theta_a(t)$ becomes $\cos t$, sinc_a becomes sinc and $\mathcal{H}\text{sinc}_a$ becomes $\mathcal{H}\text{sinc}$ which is given at $t \in \mathbb{R}$ by equation (5.9). Therefore Theorem 4.1, Theorem 5.7 and the well-known Shannon sampling theorem imply the following corollary.

Corollary 5.9 *The following statements are equivalent:*

1. *The real signal $\rho \in L^2(\mathbb{R})$ is bandlimited such that $\text{supp } \hat{\rho} \subseteq [-1, 1]$.*
2. *$\mathcal{H}(\rho(\cdot) \cos(\cdot))(t) = \rho(t) \sin t$, $t \in \mathbb{R}$.*
3. *$\mathcal{H}(\rho(\cdot) e^{i\cdot})(t) = -i\rho(t)e^{it}$, $t \in \mathbb{R}$.*
4. *$\rho(t) = \sum_{k \in \mathbb{Z}} \rho(2k\pi) \text{sinc}(t - 2k\pi) - \sum_{k \in \mathbb{Z}} \mathcal{H}\rho(2k\pi) \frac{1 - \cos(t - 2k\pi)}{t - 2k\pi}$, $t \in \mathbb{R}$.*
5. *$\rho(t) = \sum_{k \in \mathbb{Z}} \rho(k\pi) \text{sinc}(t - k\pi)$, $t \in \mathbb{R}$.*

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